table ← Joint pmfs

function ← Joint pdfs

6.3 Joint pmf and independence.

Suppose X and Y are indep. and have pmf $p_X()$ and $p_Y()$.

What is the joint pmf p_{XY}

Ex 6.40 Roll 2 fair dice: X_1 and X_2 are outcomes. we know there are indep.

$P(X_1=3, X_2=5)$

$p_{X_1}(3) = \frac{1}{6}$ $\quad$ $p_{X_2}(5) = \frac{1}{6}$ $\quad = \frac{1}{6} \cdot \frac{1}{6}$

$$p_{X_1X_2}(i,j) = \mathbb{P}(X_1 = i, X_2 = j) \quad i,j \in \{1, \dots, 6\}.$$

$$= \mathbb{P}(X_1 = i)\, \mathbb{P}(X_2 = j)$$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

If X_1 and X_2 are independent discrete rvs.

then

$$\boxed{p_{X_1X_2}(i,j) = p_{X_1}(i)\, p_{X_2}(j)}$$

Joint pmfs exist even when rvs take a countably infinite # of values.

If $X \sim \text{Geom}(p)$ range$(X) = \{1, 2, \dots\}$

$Y \sim \text{Geom}(p)$. range$(Y) = \{1, 2, \dots\}$

Find the pmf of $Z = \min(X, Y)$

$\mathbb{P}(\min(X, Y) > 3)$

$= \mathbb{P}(X > 3, Y > 3) = \mathbb{P}(X > 3)\,\mathbb{P}(Y > 3)$

$= (1-p)^3 (1-p)^3.$

Q:

Find the joint pmf of X, Y $\quad X \sim Geom(p) \quad Y \sim Geom(p)$

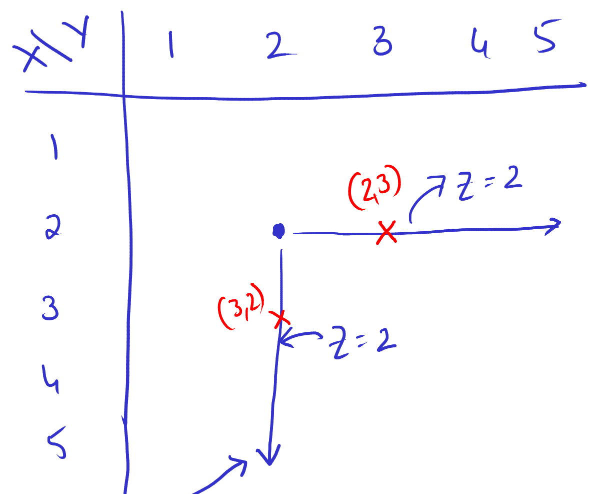
$$p_{X,Y}(i,j) = p_X(i)\, p_Y(j) = (1-p)^{i-1} p\, (1-p)^{j-1} p$$

$$p_X(i) = (1-p)^{i-1} p = p_Y(i)$$

Find pmf of $Z = \min(X,Y)$, find this using the joint pmf of X and Y.

$$p_Z(j) = P(\min(X,Y) = j)$$

$$p_Z(2) = \sum_{j=2}^{\infty} p_{X,Y}(2,j) + \sum_{i=2}^{\infty} p_{X,Y}(i,2)$$



$$p_Z(t) = \sum_{j=t}^{\infty} p_{X,Y}(t,j) + \sum_{i=t}^{\infty} p_{X,Y}(i,t)$$

$$= \sum_{j=t}^{\infty} p_X(t)\, p_Y(j) + \sum_{i=t}^{\infty} p_X(i)\, p_Y(t)$$

$$p_{XY}(i,j) = (1-p)^{i-1} p\, (1-p)^{j-1} p$$

6-3 Independence in the continuous case.

A joint pdf $f_{XY}(u,v)$ represents two independent variables X and Y if and only if

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

It decomposes into a product with x & y separated.

Ex: $$f_{XY}(x,y) = \frac{7\, e^{-x^2/2}}{\sqrt{2\pi}}\, e^{-7y}$$

Given to be a joint pdf. What are X and Y?

$$= \left(\frac{e^{-x^2/2}}{\sqrt{2\pi}}\right)\left(7e^{-7y}\right)$$

$\rightarrow$ $Y \sim Exp(7)$

$\rightarrow$ pdf of a normal $N(0,1) \sim X$

X and Y are indep. because $f_{X,Y}(x,y)$ is a product!

Can we use this to prove DEPENDENCE.

Suppose two rvs X and Y are jointly continuous and hence have a joint pdf f_{XY}.

How to prove X and Y not indep.?

Simply demonstrate x, y such that

$$\boxed{f_{XY}(x,y) \neq f_X(x)\, f_Y(y).}$$

One pair of values (x,y) st the product does not hold.

An example follows.

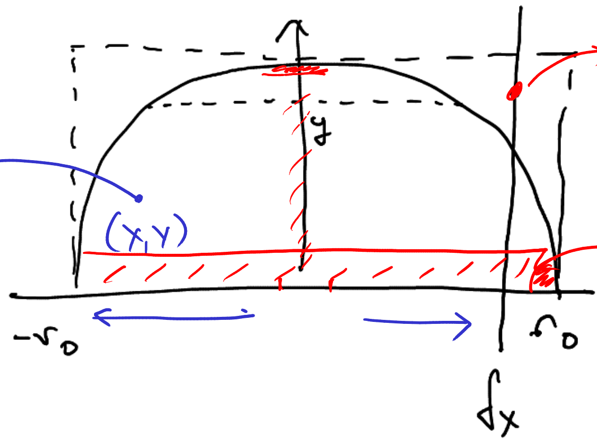
Consider our previous example where (X, Y) represents a point chosen uniformly randomly inside a semi-circle of radius r.

$$f_{X,Y}(u,v) = \begin{cases} \dfrac{1}{\text{Area}(D)} & \text{for } (u,v) \in \{(u,v) : u^2+v^2 \le r_0^2,\ u \ge 0\} \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow 0$

$f_{X,Y}(u,v) =$

Uniform on this disk.



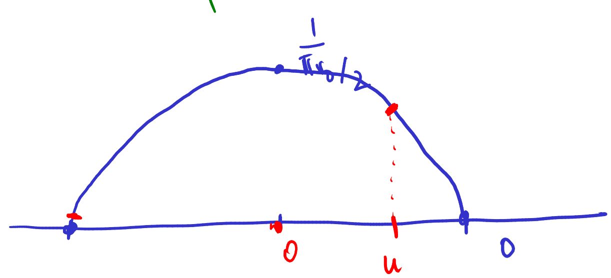
Y is small, lots of area

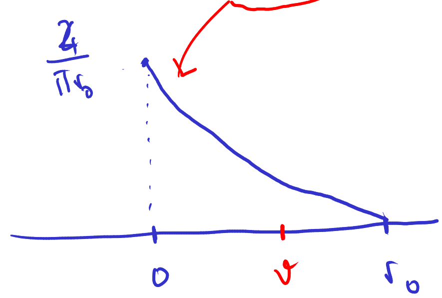
$$f_X(u) = \frac{\sqrt{r_0^2 - u^2}}{\pi r_0^2/2} \qquad -r_0 \le u \le r_0$$

Area

$$f_Y(v) = \frac{2\sqrt{r_0^2 - v^2}}{\pi r_0^2/2} \qquad 0 \le v \le r_0 \quad [0, r_0]$$

Let us plot these two marginal pdfs.





Q: Are X & Y independent

$$f_{X,Y}(u,v) = f_X(u)\, f_Y(v)$$

Now, pick a point (u,v) that is outside the semicircular disc BUT inside the open rectangle $(-r_0, r_0) \times (0, r_0)$

$$f_{X,Y}(u,v) = 0 \qquad f_X(u) > 0 \quad 0 < u < r_0$$

$$f_Y(v) > 0 \quad 0 < v < r_0$$

$$\neq f_X(u)\, f_Y(v) \implies \text{Dependendent.}$$

If $(X,Y) \sim$ Uniform (Rectangle) then X & Y are independent.

Claim: If the region is a rectangle, then the random variables are uniform.

POLL: You get phone calls from your mother and grandmother independently. Let X be the time until the next call from your mother, and Y be the time for the next call from your grandma. Suppose $X \sim \text{Exp}(\lambda)$ $Y \sim \text{Exp}(\mu)$

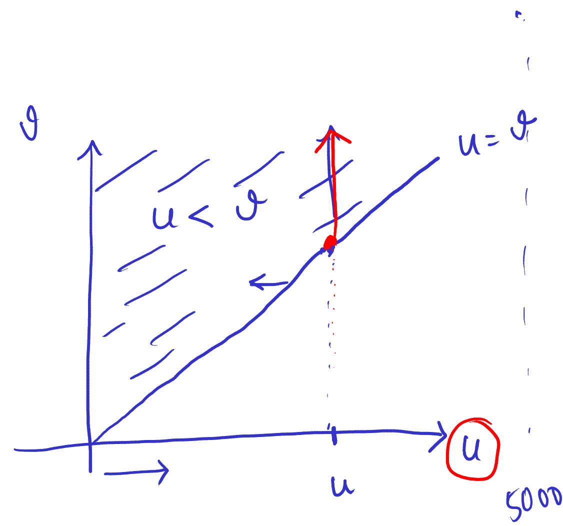
Find the probability that your mom calls before your grandma.

$$f_X(u) = \lambda e^{-\lambda u}$$

$$f_Y(v) = \mu e^{-\mu v}$$

Hint: Find $f_{X,Y}(u,v)$ & compute a prob.

https://pollev.com/arjunkrishna250



$$\overset{X}{P}(X < \overset{=Y}{Y}) =$$

$$= \int_0^\infty \int_u^\infty f_{X,Y}(u,v)\, dv\, du$$

$$= \int_0^\infty \int_u^\infty \lambda e^{-\lambda u} \mu e^{-\mu v}\, dv\, du$$

$$= \lambda\mu \int_0^\infty e^{-\lambda u} \int_u^\vartheta e^{-\mu\vartheta}\, d\vartheta \; du$$

$$\int_u^\infty e^{-\mu\vartheta}\, d\vartheta = \left.\frac{e^{-\mu\vartheta}}{-\mu}\right|_u^\infty = \frac{\cancelto{0}{e^{-\mu\infty}} - e^{-\mu u}}{-\mu} = \frac{e^{-\mu u}}{\mu}$$

$$= \lambda\cancel{\mu} \int_0^\infty e^{-\lambda u} \frac{e^{-\mu u}}{\cancel{\mu}}\, du = \lambda \int_0^\infty e^{-(\lambda+\mu)u}\, du$$

$$= \lambda \left.\frac{e^{-(\lambda+\mu)u}}{-(\lambda+\mu)}\right|_0^\infty = \frac{\lambda}{-(\lambda+\mu)}\left[\cancelto{\infty}{e^{-(\lambda+\mu)\infty}} - \cancelto{1}{e^{-(\lambda+\mu)0}} \right]$$

$$= \frac{\lambda}{\lambda+\mu}$$

GENERALITIES

If you have n-randomvariables $X_1, X_2, \cdots, X_n$:

1) All discrete, then they have a joint pmf

$n=2$

$$p_{X_1 \cdots X_n}(a_1, \cdots, a_n) = P(X_1 = a_1, \cdots, X_n = a_n)$$

$$a_i \in \text{range}(X_i)$$

2) Jointly continuous: there is a function

$f_{X_1 \cdots X_n}(a_1, \cdots, a_n)$, such that

$$P(X_1 \le t_1, \cdots, X_n \le t_n)$$
$$= \underbrace{\int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_n}}_{\to\ n=2,\ \text{double integral.}} f_{X_1 \cdots X_n}(a_1 \cdots a_n)\, da_1 \cdots da_n$$

3) Expectation.

a) $$E\left[g(X_1, \cdots, X_n)\right] = \sum_{\underbrace{a_1 \cdots a_n}_{\text{big sum over range of } X_1,\ \text{range } X_2, \cdots}} \underbrace{g(a_1, \cdots, a_n)}_{\text{value}} \underbrace{p_{X_1 \cdots X_n}(a_1, \cdots, a_n)}_{\text{pmf}}$$

when jointly continuous

b) $E\left[g(X_1, \ldots, X_n)\right] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underbrace{g(a_1, \ldots a_n)}_{\text{value.}} \underbrace{f_{X_1 \ldots X_n}(a_1, \ldots, a_n)}_{\text{pdf}} \, da_1 \cdots da_n$

4) Independence: if $X_1, \ldots, X_n$ are independent

then:

a) $\overbrace{p_{X_1 \ldots X_n}(a_1, \ldots, a_n)}^{\text{joint pmfs}} = p_{X_1}(a_1) \cdots p_{X_n}(a_n)$ = product of individual pmf.

b) $\overbrace{f_{X_1 \ldots X_n}(a_1, \ldots, a_n)}^{\text{joint pdf}} = \overbrace{f_{X_1}(a_1) \cdots f_{X_n}(a_n)}^{\text{product of individual pdf.}}$